International Journal of Theoretical Physics, Vol. 33, No. 10, 1994

Generalized Expanding and Shearing Magnetized Viscous Fluid Cosmological Model

Raj Bali[1] and Mahbub Ali[1]

Received November 30, 1993

We investigate the behavior of a magnetic field in a viscous fluid cosmological model where the expansion θ in the model is proportional to σ^1_1, the component of shear tensor σ^j_i, which leads to $A = (BC)^n$. We also assume that the shear viscosity is proportional to the rate of expansion in the model. The behavior of the model in the absence of a magnetic field and viscosity is discussed as are some other physical and geometrical aspects.

1. INTRODUCTION

It is well known that in the early stages of the universe, when neutrino decoupling occurs, matter behaves like a viscous fluid. Misner (1967, 1968) has studied the effect of viscosity on the evolution of cosmological models. The role of viscosity in cosmology is investigated by Weinburg (1971), Nightingale (1973), and Klimek (1975). Heller (1974) obtained a dust-filled viscous universe in general relativity. Belinskii and Khalatnikov (1976) investigated the effect of viscosity in cosmological evolution. Belinskii and Khalatnikov (1977) also studied the effect of viscosity in a Friedmann model in which the coefficient of viscosity is assumed to be a function of energy density. Roy and Prakash (1976, 1977) investigated viscous fluid cosmological models in which the two coefficients of viscosity are considered as constants. Banerjee and Santos (1983) obtained a Bianchi type I viscous fluid cosmological model in which the coefficients of bulk and shear viscosity are proportional to the energy density, and the fluid's shear scalar is proportional to volume expansion.

[1]Department of Mathematics, University of Rajasthan, Jaipur-302004, India.

Mohanty and Pattanaik (1991) obtained an anisotropic spatially homogeneous bulk viscous model of the universe without shear viscosity. Bali and Jain (1991) obtained a viscous fluid cosmological model filled with a stiff fluid with shear viscosity but without bulk viscosity. Banerjee *et al.* (1985) obtained a Bianchi type I viscous universe where shear and bulk viscosity coefficients are power functions of the energy density and equation of state for a stiff fluid ($p = \varepsilon$). Zel'dovich and Novikov (1971) investigated the presence of a strong magnetic field exhibited by galaxies and interstellar spaces, which gives rise to a kind of viscous effect in the fluid flow (Cowling, 1957). Misner *et al.* (1973) investigated a universe in which there is a strong magnetic field contributing to the total energy of the system. The coefficients of viscosity decrease as the universe expands.

Bali and Jain (1989) obtained an anisotropic magnetized viscous fluid cosmological model in general relativity where the free gravitational field is of Petrov type D and the coefficients of shear viscosity are proportional to the rate of expansion in the model. Banerjee and Sanyal (1986) obtained some homogeneous anisotropic cosmological models with viscous fluid and magnetic field assuming a linear relation between the matter density ε, the shear scalar σ, and the expansion scalar θ together with $p = \gamma\varepsilon$. Bali (1985) obtained an expanding and shearing magnetoviscous fluid cosmological model in which the scalar of expansion θ is proportional to σ^1_1, the component of the shear tensor σ^j_i, and the coefficient of shear viscosity is proportional to the rate of expansion θ in the model. The assumption $\theta \propto \sigma^1_1$ leads to $A = (BC)^n$. Bali (1985) obtained a magnetoviscous fluid model for $n = 1$.

In this paper we have obtained a generalized expanding and shearing magnetoviscous fluid cosmological model in general relativity for general value of n. Various physical and geometrical aspects of the model are discussed. The behavior of the model in the absence of magnetic field and viscosity is also discussed.

We consider a cylindrically symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2\,dy^2 + C\,dz^2 \tag{1.1}$$

where the metric potentials are functions of time alone. The energy-momentum tensor is taken to be the sum of the energy momentum tensors M_{ij} corresponding to the viscous fluid (Landau and Lifshitz, 1963) and E_{ij}, the electromagnetic field (Lichnerowicz, 1967), given by

$$M_{ij} = (\varepsilon + p)v_i v_j + p g_{ij} - \eta(v_{i;j} + v_{j;i} + v_j v^l v_{i;l} + v_i v^l v_{j;l}) - \left(\zeta - \frac{2}{3}\eta\right)v^l_{;l}(g_{ij} + v_i v_j) \tag{1.2}$$

and

$$E_{ij} = \bar{\mu}\left\{|h|^2\left(v_i v_j + \frac{1}{2} g_{ij}\right) - h_i h_j\right\} \tag{1.3}$$

where ε is the density, p the pressure, η and ζ are the two coefficients of viscosity, and v^i is the flow vector satisfying the equation

$$g_{ij} v^i v^j = -1 \tag{1.4}$$

where $\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector, defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \varepsilon_{ijkl} F^{kl} v^j \tag{1.5}$$

where F_{kl} is the electromagnetic field tensor and ε_{ijkl} is the Levi-Civita tensor density. A semicolon stands for covariant differentiation; we assume the coordinate to be comoving so that $v^1 = 0 = v^2 = v^3$ and $v^4 = 1/A$. We take the incident magnetic field to be in the direction of the x-axis so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = F_{13} = 0$ by virtue of (1.5). Also $F_{14} = F_{24} = F_{34} = 0$ on account of the assumption of infinite conductivity of the fluid. Hence, the only nonvanishing component of F_{ij} is F_{23}. The first set of Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{1.6}$$

leads to $F_{23} = \text{const} = H$ (say).

The field equations

$$R_i^j - \frac{1}{2} R g_i^j + \Lambda g_i^j = -8\pi T_i^j \tag{1.7}$$

for the line element (1.1) are

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_2 C_4}{BC} + \frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC}\right) - \Lambda$$
$$= 8\pi\left[p - \frac{2\eta A_4}{A^2} - \left(\zeta - \frac{2}{3}\eta\right) v^l_{;l} - \frac{H^2}{2\bar{\mu} B^2 C^2}\right] \tag{1.8}$$

$$\frac{1}{A^2}\left(-\frac{C_{44}}{C} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) - \Lambda$$
$$= 8\pi\left[p - \frac{2\eta B_4}{AB} - \left(\zeta - \frac{2}{3}\eta\right) v^l_{;l} + \frac{H^2}{2\bar{\mu} B^2 C^2}\right] \tag{1.9}$$

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B} - \frac{A_{44}}{A} + \frac{A_4^2}{A^2}\right) - \Lambda$$
$$= 8\pi\left[p - \frac{2\eta C_4}{AC} - \left(\zeta - \frac{2}{3}\eta\right) v^l_{;l} + \frac{H^2}{2\bar{\mu} B^2 C^2}\right] \tag{1.10}$$

$$\frac{1}{A^2}\left(\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right)+\Lambda$$
$$=8\pi\left(\varepsilon+\frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{1.11}$$

The subscript 4 on $A, B,$ and C denotes ordinary differentiation with respect to t.

2. SOLUTION OF THE FIELD EQUATIONS

Equations (1.8)–(1.11) are four equations in five unknowns A, B, C, ε, and p. We assume that expansion (θ) in the model is proportional to the eigenvalue σ_1^1 of the shear tensor σ_i^j. This condition leads to

$$A=(BC)^n \tag{2.1}$$

From equations (1.8)–(1.10) and (2.1) we have

$$n\left(\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{2B_4C_4}{BC}\right)-\frac{B_4C_4}{BC}-\frac{B_{44}}{B}$$
$$=16\eta\pi(BC)^n\frac{B_4}{B}-\frac{nB_4}{B}-\frac{H^2(BC)^n}{2n\bar{\mu}B^2C^2} \tag{2.2}$$

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=16\pi\eta(BC)^n\left(\frac{C_4}{C}-\frac{B_4}{B}\right) \tag{2.3}$$

From equations (2.2) and (2.3) we have

$$\frac{1}{2n-1}\frac{H^2(BC)^n}{2\bar{\mu}B^2C^2}\left(\frac{C_{44}}{C}-\frac{B_{44}}{B}\right)+\frac{C_{44}}{C}\frac{B_4}{B}=\frac{B_{44}}{B}\frac{C_4}{C}+\frac{B_4C_4}{BC}\left(\frac{C_4}{C}-\frac{B_4}{B}\right) \tag{2.4}$$

Putting $BC=\mu$ and $B/C=\nu$ in equations (2.3) and (2.4), we get

$$\left(\frac{\mu\nu_4}{\nu}\right)_4=-16\pi\eta\mu^n\left(\frac{\mu\nu_4}{\nu}\right) \tag{2.5}$$

and

$$\mu_{44}+16\pi\eta\mu^n\mu_4+\frac{16\pi H^2\mu^{2n}}{\bar{\mu}(2n-1)\mu}=0 \tag{2.6}$$

To get a determinate solution, we assume that $\eta\alpha\theta$, which leads to

$$\eta=\frac{l(n+1)\mu_4}{\mu^{n+1}} \tag{2.7}$$

where l is a proportionality constant.

Using (2.7) in (2.6), we get

$$\mu\mu_{44} + 16\pi l(n+1)\mu_4^2 + K\mu^{2n} = 0 \tag{2.8}$$

where

$$K = \frac{16\pi H^2}{\bar{\mu}(2n-1)}$$

Putting $\mu_4 = f(\mu)$ in equation (2.8), we have

$$\frac{df^2}{d\mu} + \frac{32\pi(n+1)l}{\mu} f^2 + 2K\mu^{2n-1} = 0 \tag{2.9}$$

From (2.9) we get

$$f^2 = m\mu^{-32\pi(n+1)l} - \frac{K\mu^2}{n + 16\pi(m+1)l} \tag{2.10}$$

where m is a constant of integration.

Using (2.10) in (2.5), we get

$$v = k\mu^{\beta/m^{1/2}} \left\{1 + \left(1 - \frac{K\mu^{2(\alpha+1)}}{(\alpha+n)m}\right)^{1/2}\right\}^{-\beta/(\alpha+1)} \tag{2.11}$$

where β and k are constants of integration and α is an arbitrary constant. By suitable transformations of coordinates, the metric (1.1) reduces to the form

$$dS^2 = T^{2n}\, dX^2 - \frac{T^{2n}}{f^2}\, dT^2 + T^{1+\gamma}\left\{1 + \frac{fT^\alpha}{\sqrt{m}}\right\}^{-\gamma/(\alpha+1)} dY^2 + T^{1-\gamma}\left(1 + \frac{fT^\alpha}{\sqrt{m}}\right)^{\gamma/(\alpha+1)} dZ^2 \tag{2.12}$$

where γ is an arbitrary constant and

$$f^2 = mT^{-2\alpha} - \frac{KT^2}{n+\alpha} \tag{2.13}$$

If $n = 1$, then we get the model obtained by Bali (1985).

In the absence of a magnetic field, the metric (2.12) reduces to

$$dS^2 = T^{2n}\, dX^2 - \frac{T^2}{mT^{-2\alpha}}\, dT^2 + T^{1+\gamma}2^{-\gamma/(\alpha+1)}\, dY^2 + T^{1-\gamma}2^{\gamma/(\alpha+1)}\, dZ^2 \tag{2.14}$$

Also in the absence of viscosity, the metric (2.14) reduces to

$$dS^2 = T^{2n}\, dX^2 - \frac{T^2}{m}\, dT^2 + T^{1+\gamma}2^{-\gamma}\, dY^2 + T^{1-\gamma}2^{\gamma}\, dZ^2 \tag{2.15}$$

3. SOME PHYSICAL AND GEOMETRICAL FEATURES

The pressure and density for the model (2.12) are given by

$$8\pi p = \frac{1}{T^{2n}}\left[\frac{(4n+1-\gamma^2)f^2}{4T^2} - \frac{\gamma^2 K T^\alpha f}{2(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} - \frac{\gamma^2 K^2 T^{2(\alpha+1)}}{4(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2} + \frac{1}{T^2}\left(m\alpha T^{-2\alpha} + \frac{KT^2}{n+\alpha}\right)\right] + \frac{\alpha f^2(2n-1)}{3T^{2(n+1)}} + \frac{8\pi\xi f(n+1)}{T^{n+1}} + \frac{K(2n-1)}{4T^2} - \Lambda \tag{3.1}$$

and

$$8\pi\varepsilon = \frac{1}{T^{2n}}\left[\frac{(4n+1-\gamma^2)f^2}{4T^2} - \frac{\gamma^2 K T^\alpha f}{2(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} - \frac{\gamma^2 K^2 T^{2(\alpha+1)}}{4(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})}\right] - \frac{K(2n-1)}{4T^2} + \Lambda \tag{3.2}$$

where $f = [mT^{-2\alpha} - KT^2/(n+\alpha)]^{1/2}$.

The model has to satisfy the reality conditions:

(i) $p > 0$.
(ii) $\varepsilon - p \geq 0$.

Condition (i) leads to

$$\frac{(4n+1-\gamma^2)f^2}{4T^2} - \frac{\gamma^2 K T^\alpha f}{2(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} - \frac{\gamma^2 K^2 T^{2(\alpha+1)}}{4(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2} + \frac{1}{T^2}\left(mT^{-2\alpha} + \frac{KT^2}{n+\alpha}\right) + \frac{\alpha(2n-1)f^2}{3T^2} + \frac{8\pi(n+1)f}{T^{1-n}} + \frac{K(2n-1)}{4T^{2-2n}} - \Lambda T^{2n} > 0 \tag{3.3}$$

while condition (ii) yields

$$\frac{1}{T^{2n+2}}\left(m\alpha T^{-2\alpha} + \frac{KT^2}{n+\alpha}\right) + \frac{\alpha(2n-1)f^2}{3T^{2n+2}} + \frac{8\pi(n+1)f}{T^{n+1}} - 2\Lambda \leq 0 \tag{3.4}$$

which imposes a condition on Λ.

The scalar of expansion, θ calculated for the flow vector v^i is given by

$$\theta = \frac{(n+1)f}{T^{n+1}} \tag{3.5}$$

The rotation ω is identically zero and the shear is given by

$$\sigma^2 = \frac{1}{4T^{2n}}\left[\frac{f^2(4n^2-4n+1+3\gamma^2)}{3T^2} + \frac{\gamma^2 K^2 T^{2(\alpha+1)}}{(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2} + \frac{2\gamma^2 KT^\alpha f}{(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})}\right] \tag{3.6}$$

The nonvanishing components of the conformal curvature tensors are

$$C^{12}_{12} = \frac{1}{6T^{2n}}\left[\frac{(1-\gamma^2-2n-6n\gamma)f^2}{2T^2} - \frac{[m\alpha T^{-2\alpha}+KT^2/(n+\alpha)](2n+3\gamma-1)}{2T^2} + \frac{\gamma KfT^\alpha(3\alpha+6-6n-2\gamma)}{2(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} + \frac{\gamma K^2T^{2(\alpha+1)}(6\alpha+6-2\gamma)}{4(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2}\right] \tag{3.7}$$

$$C^{13}_{13} = \frac{1}{6T^{2n}}\left[\frac{(1-\gamma^2-2n+6n\gamma)f^2}{2T^2} - \frac{[m\alpha T^{-2\alpha}+KT^2/(n+\alpha)](2n-3\gamma-1)}{2T^2} - \frac{\gamma KfT^\alpha(6+3\alpha+2\gamma-6n)}{2(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} - \frac{1}{4}\frac{\gamma K^2T^{2(\alpha+1)}(6\alpha+6+2\gamma)}{(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})}\right] \tag{3.8}$$

$$C^{23}_{23} = \frac{1}{6T^{2n}}\left[\frac{(2n-1+\gamma^2)f^2}{T^2} - \frac{[m\alpha T^{-2\alpha}+KT^2/(n+\alpha)](1-2n)}{T^2} + \frac{2\gamma^2 KfT^\alpha}{(\alpha+n)\sqrt{m}\,(1+fT^\alpha/\sqrt{m})} + \frac{\gamma^2K^2T^{2(\alpha+1)}}{(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2}\right] \tag{3.9}$$

Hence, the space-time is of nondegenerate Petrov type I in general. For large values of T, the space-time is conformally flat.

The expression for

$$\frac{E^4_4}{\varepsilon} = \frac{\text{magnetic energy}}{\text{material energy}}$$

is

$$\frac{E^4_4}{\varepsilon} = (2n-1)KT^{2n} \times \left\{4\left[\frac{(4n+1-\gamma^2)f^2}{T^2} - \frac{\gamma^2 KT^{\alpha+2}}{2(\alpha+n)\sqrt{m}\,(q+fT^\alpha/\sqrt{m})} - \frac{\gamma^2K^2T^{2\alpha+4}}{4(\alpha+n)^2 m(1+fT^\alpha/\sqrt{m})^2} - \frac{K(2n-1)T^{2n}}{4} + \Lambda T^{2n+2}\right]\right\}^{-1} \tag{3.10}$$

Since $\lim_{T\to 0} E^4_4/\varepsilon \to 0$, the material energy is more dominant than the magnetic energy near a singularity. There is a singularity in the model (2.12) at $T=0$, which may be explained as a matter distribution along the axis which carries an electric current producing the transverse magnetic field. The model (2.12) starts expanding at $T=0$, goes on expanding indefinitely, and the expansion in the model stops at $T=\infty$. The nonvanishing components of the conformal curvature tensor for perfect fluid distributions are given by

$$C^{12}_{12}=\frac{1}{12T^{2n+2}}[m(1-\gamma^2-2n-6n\gamma)] \tag{3.11}$$

$$C^{13}_{13}=\frac{1}{12T^{2n+2}}[m(1-\gamma^2-2n+6n\gamma)] \tag{3.12}$$

$$C^{23}_{23}=\frac{1}{6T^{2n+2}}[m(2n-1)+\gamma^2)] \tag{3.13}$$

Hence, the space-time is of Petrov type D for $\gamma=0$ and $n=0$ also, and of nondegenerate Petrov type I otherwise.

Since $\lim_{T\to\infty}\sigma/\theta\neq 0$, the model does not approach isotropy for large values of T.

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